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A Note on User Cost and Taxes in ADAM and DREAM

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Abstract:

Based on the dynamic maximization problem of a representative firm, we derive the user cost of capital in the case of pure debt finance, and pure finance through retained earnings respectively. By doing this it is possible to determine when the user cost expression adopted in ADAM is equivalent to the user cost expression used in DREAM. The analysis encompasses some novel features. First of all we derive a general expression for the liquidation value of the firm, which we use as the firms debt restriction in the case of pure debt finance. Secondly, this allows us to discuss how the user cost of capital changes, given pure debt finance, when a stricter restriction on the maximum amount of debt is imposed.

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1 Introduction

Initiating from the seminal work of Jorgenson (1963), a large literature has investigated the user cost of capital. In recent years the focus of research has mainly been on the effect of taxes and inflation on the user cost of capital. As for the present paper, the focus is solely on the former. Specifically we are concerned with how the user cost of capital is affected by the interplay between taxes and det preferred choice of finance: debt or retained earnings.[1] To accomplish this, we consider the dynamic maximization problem of a representative firm. The firm is assumed to maximize it's market value, while being subject to Danish tax laws. The analysis is preformed in continous time, taxes and the interest rate are assumed constant. The specifics are presented below. We start out by presenting the basic set up in section 2.1. In section 2.2 the maximization problem is subsequently solved, and user costs are derived for three cases: pure debt finance, finance through retained earnings, and an intermediate case, where the firm is indifferent between the use of debt or retained earnings. Section 3 is reserved for discussion and concluding remarks.

2 Analysis

2.1 Set-up

The value for the firm is determined by an arbitrage condition which states, that the after tax gain form investing in shares must equal the after tax yield on bonds

$$(1-t_r)\, iV(t) = (1-t_g)\, \dot{V}(t) + (1-t_d)\, D(t)\,. \tag{1}$$

t_r, t_g, t_d denotes taxes on interest income, capital gains and dividends respectively. $V(t)$ is the value of the firm, and $D(t)$ are dividends . i is the rate of interest on bonds. Integrating (1) yield:

$$V_0 = \frac{1-t_d}{1-t_g} \int_0^{\infty} D(u) e^{-\frac{(1-t_r)i}{(1-t_g)}u} du. \tag{2}$$

(2) implies, that the maximization of dividends for the owners, are equivalent to maximizing the present value of the company. By definition, dividends are given by

$$\begin{aligned} D(t) &= (1-t_c)\left(p(t)\, F(K(t), L(t)) - w(t)L(t) - iB(t)\right) \\ &\quad + t_c \hat{\delta} A(t) - q(t)\, I(t) + b(t) \end{aligned} \tag{3}$$

[1] We abstract from the case where firms finance their investments through new shares issuing.

where t_c is the corporate tax, p is the price of output, $F(K, L)$ is the constant returns technology available to the firm, K is the stock of capital, and L denotes the labour input. w is the real wage, B is the stock of debt at time t, $\hat{\delta}$ is the rate of depreciation allowances, A is the book value of the capital stock, q is the price on new capital, I denotes investments and finally b denotes the amount of borrowing during the time interval t to $t + \Delta t$. As usual, capital is accumulated according to

$$\dot{K}(t) = I(t) - \delta K(t) \tag{4}$$

where δ accordingly represent the rate of (true) economic depreciation. The book value of the capital stock evolves in accordance with (5) :

$$\dot{A}(t) = q(t) I(t) - \hat{\delta} A(t) \tag{5}$$

Additionally the stock of (gross) debt rises with borrowing:[2]

$$\dot{B}(t) = b(t). \tag{6}$$

Furthermore, based on derivations along the lines of Hayashi (1982), we impose the following restricitons on the maximum amount of debt that a firm can obtain

$$K(t) \frac{\beta(t)(1-t_g)}{1-t_d} + \frac{\gamma(t)(1-t_g)}{1-t_d} A(t) \geq B(t) \tag{7}$$

and

$$B(t) \geq 0. \tag{8}$$

(7) states, that total debts are not allowed to rise above a weighted sum of the capital stock and its book value, where the weights are the shadow price on the respective capital stocks corrected for taxes on capital gains and dividends. We impose the restriction (7) so as to capture, that a potential creditor is partial to supplying the firm with funds only in the presence of adequate collateral. Given this, (7) simply defines "adequate" collateral, as the liquidation value of the firm. That (7) in fact captures this is proven in appendix A.[3] In section 3 we discuss the implications of relaxing this assumption.

The restriction (8) ensures, that the firm's debt cannot be negative. This might not be an innocent assumption since it could have some influence on the user cost expression in the case of retained earnings. In principle, one could impose a minimum restrictions on dividend pay-outs in stead of (8). Either way, the point is that we abstract from the case where the firm could act as a financial intermediary.

[2] Remember that expences for interests are accounted for in (3).

[3] In the stationary state – and given pure debt finance - (7)colapses to the somewhat neater expression:

$$(1-z) qK + zA = B.$$

where $z = \frac{t_c \hat{\delta}}{(1-t_c)i + \hat{\delta}}$. See appendix C for details.

2.2 Solving the model

The firm's problem is to choose a sequence $\{I, b, K, A, B\}_0^\infty$ so as to maximize (2). The (current value) Hamiltonian connected to the problem can be stated[4]

$$
\begin{aligned}
& H(I, b, K, L, A, B, \alpha, \beta, \gamma, \lambda, \eta) \\
= \; & \frac{1-t_d}{1-t_g}\left[(1-t_c)(pF(K,L) - wL - iB) + t_c\hat{\delta}A - qI + b\right] \\
& +\alpha b + \beta(I - \delta K) + \gamma\left(qI - \hat{\delta}A\right) + \lambda\left(K\frac{\beta}{\frac{1-t_d}{1-t_g}} + \frac{\gamma}{\frac{1-t_d}{1-t_g}}A - B\right) + \eta B
\end{aligned}
$$

The first order conditions for the state variables L, I and for the controls K, A,B are derived in the appendix. In what follows, we limit our attention to the stationary state.

In the absence of convex costs of installation, investments are no longer determined by the first order conditions. In stead they are determined through the controls A, K, so as to ensure that the direct cost of investments $q\frac{1-t_d}{1-t_g}$ equals marginal benefit. The marginal benefit can be decomposed into the value of increasing the stock of capital by one unit (β) and the value in terms of the increase in the book value of the capital stock $q\gamma$:

$$
H_I : q\frac{1-t_d}{1-t_g} = \beta + q\gamma. \tag{9}
$$

As for the intensity of borrowing, this too is regulated indirectly through the relevant controls, G, K. In optimum, the shadow price on borrowing, $-\alpha$, will equal the marginal benefit to increased borrowing in the shape of disposable dividends $\frac{1-t_d}{1-t_g}$:

$$
H_L : \frac{1-t_d}{1-t_g} = -\alpha. \tag{10}
$$

Furthermore, the shadow price of increasing the book value, is determined by the discounted value of depreciation allowances, i.e.

$$
H_A : \gamma = \frac{\left(\frac{1-t_d}{1-t_g}\right)t_c\hat{\delta}}{\left(\frac{(1-t_r)i}{(1-t_g)} + \hat{\delta} - \frac{\lambda}{\frac{(1-t_r)i}{(1-t_g)}}\right)}. \tag{11}
$$

λ reflects whether the Kuhn-Tucker condition (7) is binding or not. Hence, the rate of discounting, when it comes to the depreciation allowances, depends on which "regime" we are considering; debt finance or retained earnings.

The stock of debt is implicitly determined by

$$
H_B : \lambda - \eta = \left(\frac{1-t_d}{1-t_g}\right)\left(\frac{(1-t_r)\,i}{(1-t_g)} - (1-t_c)\,i\right). \tag{12}
$$

[4]Time subscripts are suppressed from now on.

The right hand side express the cost differential between raising funds for investments through retained earnings $\left(\frac{(1-t_r)i}{(1-t_g)}\right)$ or debt $((1-t_c)\,i)$ respectively. Obviously, both Kuhn-Tucker conditions cannot be binding simultaneously (i.e. both λ and η cannot be positive at the same time), since that would imply that the ratio of debt to liquidation value is 1, and that the stock of debt is zero. Hence, we are left with three cases $(\lambda > 0, \eta = 0)$, $(\lambda = 0, \eta > 0)$ and $(\lambda = 0, \eta = 0)$; i.e. full debt finance, financing solely through retained earnings and an intermediate case, where the firm is indifferent between retained earnings and debt as the source of finance. This last situation arises if the cost of the different schemes of finance are identical: $\frac{(1-t_r)i}{(1-t_g)} = (1-t_c)$. Hence, in this case, neither (7) nor (8) is binding.

Finally we have the first order condition with respect to the stock of capital

$$H_K : (1-t_c)\,pF_K = q\left(1 - \frac{t_c\hat{\delta}}{\left(\frac{(1-t_r)i}{(1-t_g)} + \hat{\delta} - \frac{\lambda}{\frac{1-t_d}{1-t_g}}\right)}\right)\left(\frac{(1-t_r)\,i}{(1-t_g)} + \delta - \frac{\lambda}{\frac{1-t_d}{1-t_g}}\right).$$

The left hand side represents the required return on capital investments net of taxes. The right hand side is the marginal cost associated with investments. These costs reflect the direct (effective) cost of investing, $q\left(1 - \frac{t_c\hat{\delta}}{\frac{(1-t_r)i}{(1-t_g)} + \hat{\delta} - \frac{\lambda}{\frac{1-t_d}{1-t_g}}}\right)$, the opportunity costs of investing, $\frac{(1-t_r)i}{(1-t_g)} + \delta$, and finally the cost of raising the funds for investing, i.e. the cost of finance, $\frac{\lambda}{\frac{1-t_d}{1-t_g}}$. Evidently user cost depend on whether (7) or (8) is binding or not. It can therefore be shown (see appendix B), that there are three possible user cost expressions depending on which of the Kuhn-Tucker conditions (if any) hold.

CASE I: Pure debt finance $\left(\lambda > 0, \eta = 0; \frac{(1-t_r)i}{(1-t_g)} > (1-t_c)\,i\right)$

$$pF_K = \frac{q}{(1-t_c)}\left(1 - \frac{t_c\hat{\delta}}{(1-t_c)\,i + \hat{\delta}}\right)(\delta + (1-t_c)\,i) \quad (13)$$

The left hand side is the required (pre-tax) marginal yield on capital investments, and the right hand side equals user costs given pure debt finance. The user costs consists of the cost of capital net of depreciation allowances , $q\left(1 - \frac{t_c\hat{\delta}}{(1-t_c)i+\hat{\delta}}\right)$, the (opportunity) cost of funds $i\,(1-t_c)$ and of the costs from depreciation of existing capital stock δ. The last term $\frac{1}{(1-t_c)}$ is present due to taxation of corporate revenue. To examine the effect of depreciation allowances more carefully, on can rewrite (13) so as to yield

$$pF_K = q\left(i + \delta - (1-t_c)\,i\left(\frac{\hat{\delta} - \delta}{(1-t_c)\,i + \hat{\delta}}\right)\right) \quad (14)$$

If $\hat{\delta} - \delta > 0$ this expression tend to reduce user cost. Notice, that if $\hat{\delta} = \delta$ then user costs collapses to $q(i + \delta)$, and hence is independent of corporate taxes. To understand why this is the case, remember that the firm finances it's investments solely through borrowing, and that *all* rents are absorbed by the servicing of its debt obligations (this was insured by using the restriction (7)). Profits, Π, are generally defined as total revenues, $F(K, L)$, less total cost, i.e.

$$\Pi = F(K, L) - uK - wL,$$

where $u = i + \delta$.[5] Interest payments and labor expenses are tax deductible, and so is the cost due to depreciation as $\hat{\delta} = \delta$. Therefore after tax profits must be equal to:

$$\Pi (1 - t_c) = (1 - t_c) (F(K, L) - uK - wL). \tag{15}$$

Hence, as (15) states, output and *all* inputs are taxed at the same rate. This is a neat result, because it implies that corporate taxes are, in this specific scenario, equivalent to a tax on *profits*. Hence, the investment behavior of the firm is *completely* unaffected by t_c.[6] This is the intuition behind the user cost expression (14) when $\hat{\delta} = \delta$.

In Denmark, however, depreciation allowances are 30% on machinery which undoubtedly exceeds the rate of true economic depreciation, a situation often labeled *accelerated depreciation.* Under these circumstances (i.e. pure debt finance and $\hat{\delta} > \delta$) corporate taxes will tend to *reduce* user cost, a phenomena sometimes referred to as "the taxation paradox". The intuition behind this result is quite simple in the light of the previous considerations. As noted, the corporate tax is neutral with respect the cost of capital if $\hat{\delta} = \delta$. If $\hat{\delta} > \delta$, however, increasing taxes makes larger tax deductions on capital investments viable. If the rate of depreciation allowances exceeds true economic depreciation, the government therefore effectively subsidizes capital investments in the production sector. Hence the result: increasing taxes tends to increase investments by lowering the user cost of capital.[7] This is what the term $(1 - t_c) i \left(\frac{\delta - \hat{\delta}}{(1 - t_c) i + \hat{\delta}} \right)$ reflects.

If profits are present, however, corporate taxes are no longer (equivalent to) a tax on profits. The reason is that dividends are not tax deductible. Thus user costs will no longer be independent of personal taxes as in the case above. This brings us to the case of retained earnings.

[5] Hence we are, for the sake of simplicity, considering the one good situation, i.e. $p = q \equiv 1$.

[6] This insight is due to Stiglitz (1973).

[7] To prove this, differentiate (14) with respect to t_c :

$$-i \left(\frac{\hat{\delta} - \delta}{(1 - t_c) i + \hat{\delta}} \right) \left(1 - \frac{(1 - t_c) i}{\left((1 - t_c) i + \hat{\delta} \right)} \right) < 0$$

iff $\hat{\delta} - \delta > 0$.

CASE II : Retained earnings $\left(\lambda = 0, \eta > 0; \frac{(1-t_r)i}{(1-t_g)} < (1-t_c)\, i\right)$ In this case, as shown in the appendix, user costs are:

$$(1-t_c)\, pF_K = q\left(1 - \frac{t_c\hat{\delta}}{\left(\frac{i(1-t_r)}{(1-t_g)} + \hat{\delta}\right)}\right)\left(\frac{i(1-t_r)}{(1-t_g)} + \delta\right). \tag{16}$$

As above, the term $q\left(1 - \frac{t_c\hat{\delta}}{\frac{(1-t_r)i}{(1-t_g)} + \hat{\delta}}\right)$ can be interpreted as the effective price on capital, adjusted for tax allowances on depreciation. $\frac{i(1-t_r)}{(1-t_g)}$ reflects the opportunity costs of funds in the present scenario, and δ the costs due to the wear and tear on capital equipment. To understand why $\frac{(1-t_r)i}{(1-t_g)}$ is the relevant discount factor in the present scenario, remember that dividends (by way of construction) are non-zero. Therefore the costs of capital reflects the opportunity costs of retained earnings, as viewed by the stockholders. Now, the income tax rate $(1-t_r)$ tends to lower the opportunity costs of capital, since 1 kr. of retained earnings only is valued at, say, 50 øre to the stockholder, due to personal taxation (if $t_r = 0.5$). At the same time, however, retained earnings tend to increase the value of the company's stock, which is taxed (upon realization), at the rate $(1-t_g)$. The tax on capital gains must therefore tend to *increase* the opportunity costs of capital. This explains why a different rate of discounting is relevant in the case of retained earnings vis-á-vis the case of pure debt finance. Finally notice, that increasing corporate taxes *always* increase user cost under retained earnings. Hence, the taxation paradox does *not* carry over to the case of retained earnings.[8]

CASE III: The intermediate case $\left(\lambda = 0, \eta = 0; \frac{(1-t_r)i}{(1-t_g)} = (1-t_c)\, i\right)$ In this case the firm is indifferent as to which source of finance should be used. Aside from this the interpretation of the expression is as above since either of the two expressions could be used to compute the cost of capital.[9]

3 Discussion and conclusion

The present paper has explored the derivation of the user cost of capital, when firm maximizes it's stock market value. The user cost expression

[8] To prove this, one can show that:

$$\frac{\partial u}{\partial t_c} = \left(\frac{q}{1-t_c}\right)\left(\frac{\frac{(1-t_r)i}{(1-t_g)} + \delta}{\frac{(1-t_r)i}{(1-t_g)} + \hat{\delta}}\right)\left(\frac{\frac{(1-t_r)i}{(1-t_g)}}{1-t_c}\left(\hat{\delta} - \delta\right)\right) > 0,$$

where u is the user cost expression presented in the text.

[9] This result essentially reflects the Modigliani-Miller Theorem, which states, that absent distorting taxes firm's are indifferent between the source of finance. If $\frac{1-t_r}{1-t_g} = 1 - t_c$ taxation is non-distortionary with respect to the source of finance.

used in ADAM (evaluated in the stationary state) is exactly equal to (13).[10] Hence, as shown above, the firm's in ADAM finance their investments entirely through debt. Furthermore, and this too is crucial, *all* revenue must be used for the servicing of these debt obligations, as no personal taxrates are present in the expression.[11] Suppose this latter condition isn't fulfilled. Specifically, assume that the firm faces a restriction on debts which prohibits it from borrowing its full discounted present value. In a broader context this is possibly the more plausible case, since its difficult to determine exactly how much a (real life) company is worth. In the presence of accelerated depreciation $qK \geq B$ would do the job. Going through the deriviations above, one will obtain the following user cost expression for the case of debt finance

$$pF_K = \frac{q}{(1-t_c)}\left((1-t_c)(i+\delta) - \frac{t_c(1-t_r)i}{(1-t_g)}\left(\frac{\hat{\delta}-\delta}{\frac{(1-t_r)i}{(1-t_g)}+\hat{\delta}}\right)\right). \tag{17}$$

Notice that user cost isn't independent of t_r, t_g any longer. The reason is, that the revenue flowing to the firm, and which is not used to service debt, has to be distributed to the owners, i.e. the shareholders. The point is, that (exactly) *all* revenue has to be used for debt service if the user cost expression (13) is to arise.[12]

In DREAM profits, net of taxes, are present and dividends are therefore positive. Hence, personal taxes matter. The exact user cost expression can be viewed as a convex combination of (17) and (16).[13] The weight assigned equals the share of investments financed by debt and retained earnings.[14]

Aside from the convex costs of installation, this is the fundamental difference between user cost in DREAM, and user cost in ADAM, evaluated in stationary state.

[10] See Dam (1995).

[11] The reader might wonder whether the taxation paradox arises in ADAM. The answer is, contrary to what one would expect, in the negative. The reason is identified in Dalgaard (1998) and has to do with the precise modelling of the discounted value of depreciation allowances.

[12] Notice, however, that the taxation paradox still arises in this scenario.

[13] See Knudsen et al (1998).

[14] At present it is assumed that 40% of all investments are financed through debt.

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A The general debt restriction

We start out by stating the maximization problem without the debt restriction. Based on this we derive the "needed" restriction on debt. The problem is

$$\begin{aligned} MAX\ V_0 &= \phi \int_0^\infty De^{-Ru} du, \\ \phi &= \frac{1-t_d}{1-t_g}, R = \frac{i(1-t_r)}{(1-t_g)}. \end{aligned}$$

$$I, L \gtreqless 0$$

$$\dot{B} = b$$

$$\dot{K} = I - \delta K$$

$$\dot{A} = qI - \hat{\delta} A$$

and three terminal conditions on the state variables:

$$\begin{aligned} \lim_{t\to\infty} Ae^{-Rt} &= 0 \\ \lim_{t\to\infty} Ke^{-Rt} &= 0 \\ \lim_{t\to\infty} Be^{-Rt} &= 0 \end{aligned}$$

The Hamiltonian:

$$\begin{aligned} & H(I, b, K, L, A, B, \alpha, \beta, \gamma) \\ = \ & \phi \left[(1-t_c)(pF(K,L) - wL - iB) + t_c \hat{\delta} A - qI + b \right] \\ & + \alpha b + \beta (I - \delta K) + \gamma \left(qI - \hat{\delta} A \right) \end{aligned}$$

First order conditions:

$$H_L : pF_L = w.$$

$$\begin{aligned} H_I &: -q\phi + \beta + \gamma q = 0 \qquad (18) \\ & \Updownarrow \\ \frac{\beta}{\phi - \gamma} &= q. \end{aligned}$$

$$H_b : \phi + \alpha = 0 \qquad (19)$$

$$H_K : \phi (1-t_c) pF_K - \beta\delta = -\dot{\beta} + \beta R \qquad (20)$$

$$H_A : \phi t_c \hat{\delta} - \gamma \hat{\delta} = -\dot{\gamma} + \gamma R \qquad (21)$$

$$H_B : -\phi (1-t_c) i = -\dot{\alpha} + \alpha R \qquad (22)$$

Now we postulate that

$$\gamma_0 A_0+\beta_0 K_0+\alpha_0 B_0 = \phi \int_0^\infty \left((1-t_c)\left(pF(K,L)-wL-iB\right)+t_c\hat{\delta}A-qI+b\right)e^{-Ru}du$$

Turning towards the first integral we apply The Euler Theorem and use the first order condition with respect to labor. This yields (where we use that $b=\dot{B}$):

$$\phi\int_0^\infty \left((1-t_c)\,pF_K K+t_c\hat{\delta}A-qI-(1-t_c)\,iB+\dot{B}\right)e^{-Ru}du \tag{23}$$

Next, we consider the transversality conditions

$$\lim_{t\to\infty}\alpha B e^{-Rt}=0$$

$$\begin{aligned}\lim_{t\to\infty}\gamma A e^{-Rt} &= 0\\ \lim_{t\to\infty}\beta K e^{-Rt} &= 0\end{aligned}$$

Following the idea of the proof of proposition 1 in Hayashi (1982) we consider

$$\begin{aligned}&\frac{d}{dt}\gamma A e^{-Rt}+\frac{d}{dt}\beta K e^{-Rt}+\frac{d}{dt}\alpha B e^{-Rt}\\ =\;&\left(\dot{\gamma}A+\gamma\dot{A}-R\gamma A\right)e^{-Rt}+\left(\dot{\beta}K+\beta\dot{K}-R\beta K\right)e^{-Rt}\\ &+\left(\dot{\alpha}B+\alpha\dot{B}-R\alpha B\right)e^{-Rt}\end{aligned}$$

Inserting the first order conditions (i.e.$\dot{\gamma},\dot{\beta},\dot{\alpha}$) and the identities $\dot{A}=qI-\hat{\delta}A$, $\dot{K}=I-\delta K$:

$$\begin{aligned}&\left[\left(-\phi t_c\hat{\delta}+\gamma\left(R+\hat{\delta}\right)\right)A+\gamma\left(qI-\hat{\delta}A\right)-R\gamma A\right]e^{-Rt}+\\ &\left[\left(-\phi\left(1-t_c\right)pF_K+\beta\left(R+\delta\right)\right)K+\beta\left(I-\delta K\right)-R\beta K\right]e^{-Rt}+\\ &\left[\left(\phi\left(1-t_c\right)i+\alpha R\right)B-\phi\dot{B}-R\alpha B\right]e^{-Rt}\end{aligned}$$

$$\begin{aligned}=\;&\left[-\phi t_c\hat{\delta}A+\gamma qI\right]e^{-Rt}+\left[-\phi\left(1-t_c\right)pF_K K+\beta I\right]e^{-Rt}\\ &+\left[\phi\left(1-t_c\right)iB-\phi\dot{B}\right]e^{-Rt}\\ =\;&\left(-\phi t_c\hat{\delta}A+\gamma qI+\left[-\phi\left(1-t_c\right)pF_K K+\beta I\right]+\phi\left(1-t_c\right)iB-\phi\dot{B}\right)e^{-Rt}\end{aligned}$$

Using the first order condition with respect to investments implies that β may be written as

$$\beta=q\left(\phi-\gamma\right)$$

Inserting this above yields

$$\begin{aligned}&\left(-\phi t_c\hat{\delta}A+\gamma qI+\left[-\phi\left(1-t_c\right)pF_K K+q\left(\phi-\gamma\right)I\right]+\phi\left(1-t_c\right)iB-\phi\dot{B}\right)e^{-Rt}\\ =\;&-\phi\left(t_c\hat{\delta}A+\left(1-t_c\right)pF_K K-qI-\phi\left(1-t_c\right)iB+\phi\dot{B}\right)e^{-Rt}\end{aligned}$$

Thus we have shown that

$$\frac{d}{dt}\gamma Ae^{-Rt}+\frac{d}{dt}\beta Ke^{-Rt}+\frac{d}{dt}\alpha Be^{-Rt}=\phi\left(\left(1-t_c\right)pF_KK+t_c\hat{\delta}A-qI-\left(1-t_c\right)iB+\dot{B}\right)e^{-Ru} \tag{24}$$

Inserting (24) into the object function (23) yields

$$\begin{aligned}&\phi\int_0^\infty\left(\left(1-t_c\right)pF_KK+t_c\hat{\delta}A-qI-\left(1-t_c\right)iB+\dot{B}\right)e^{-Ru}\\ =&-\int_0^\infty\frac{d}{dt}\gamma Ae^{-Rt}+\frac{d}{dt}\beta Ke^{-Rt}dt+\frac{d}{dt}\alpha Be^{-Rt}dt\end{aligned} \tag{25}$$

Finally observe that from the transversality condition with respect to K, we have that

$$\left[\beta Ke^{-Rt}\right]_0^\infty=-\beta_0K_0$$

and similarly for the book value of capital, A

$$\left[\gamma Ae^{-Rt}\right]_0^\infty=-\gamma_0A_0$$

$$\left[\alpha Be^{-Rt}\right]_0^\infty=-\alpha_0B_0$$

Inserting these expressions in (25)

$$\phi\int_0^\infty\left(\left(1-t_c\right)pF_KK+t_c\hat{\delta}A-qI-\left(1-t_c\right)iB+\dot{B}\right)e^{-Ru}du=\gamma_0A_0+\beta_0K_0+\alpha_0B_0$$

Which implies that

$$\phi\left(\frac{\gamma_0}{\phi}A_0+\frac{\beta_0}{\phi}K_0+\frac{\alpha_0}{\phi}B_0\right)=V_0=\phi\int_0^\infty De^{-Ru}du.$$

Additionally we have from H_b that $\alpha=-\phi$ why:

$$\phi\left(\frac{\gamma_0}{\phi}A_0+\frac{\beta_0}{\phi}K_0-B_0\right)=V_0=\phi\int_0^\infty De^{-Ru}du.$$

Thus we have shown that the restriction

$$B_0=\frac{\gamma_0}{\phi}A_0+\frac{\beta_0}{\phi}K_0\quad\Rightarrow\quad\int_0^\infty De^{-Ru}du=0$$

B The firm's problem

The problem as stated in the text

$$\begin{aligned}MAX\ V_0&=\phi\int_0^\infty De^{-Ru}du,\\ \phi&=\frac{1-t_d}{1-t_g},R=\frac{i(1-t_r)}{(1-t_g)}.\end{aligned}$$

$$I, L \gtreqless 0$$

$$\dot{B} = b$$

$$\dot{K} = I - \delta K$$

$$\dot{A} = qI - \hat{\delta} A$$

$$K\frac{\beta}{\phi} + \frac{\gamma}{\phi}A - B \geq 0$$

$$B \geq 0$$

The Hamiltonian:

$$\begin{aligned} & H\left(I, b, K, A, B, \alpha, \beta, \gamma, \lambda, \eta\right) \\ = \ & \phi\left[\left(1 - t_c\right)\left(pF(K) - iB\right) + t_c\hat{\delta}A - qI + b\right] \\ & + \alpha b + \beta\left(I - \delta K\right) + \gamma\left(qI - \hat{\delta}A\right) + \lambda\left(K\frac{\beta}{\phi} + \frac{\gamma}{\phi}A - B\right) + \eta B \end{aligned}$$

First order conditions:

$$\begin{aligned} H_I &: -q\phi + \beta + \gamma q = 0 \qquad (26) \\ & \Updownarrow \\ \frac{\beta}{\phi - \gamma} &= q. \end{aligned}$$

$$H_b : \phi + \alpha = 0 \qquad (27)$$

$$H_K : \phi\left(1 - t_c\right)pF_K - \beta\delta + \frac{\lambda}{\phi}\beta = -\dot{\beta} + \beta R \qquad (28)$$

$$H_A : \phi t_c\hat{\delta} - \gamma\hat{\delta} + \frac{\lambda}{\phi}\gamma = -\dot{\gamma} + \gamma R \qquad (29)$$

$$H_B : -\phi\left(1 - t_c\right)i - \lambda + \eta = -\dot{\alpha} + \alpha R \qquad (30)$$

the Kuhn-Tucker conditions:

$$\lambda\left(K\frac{\beta}{\phi} + \frac{\gamma}{\phi}A - B\right) = 0, \ \lambda \geq 0, \qquad (31)$$

$$\eta B = 0, \ \eta \geq 0, \qquad (32)$$

Now, we are interested in investigating under which circumstances (31) and (32) are binding, and in what that means for the cost of capital.

Assume stationarity:

$$\dot{\beta} = \dot{\gamma} = \dot{\alpha} = 0 \qquad (33)$$

By (27)

$$-\phi = \alpha \qquad (34)$$

From (29):

$$\begin{aligned} \phi t_c \hat{\delta} &= \gamma \left(R + \hat{\delta} - \frac{\lambda}{\phi} \right) \\ &\Updownarrow \\ \gamma &= \frac{\phi t_c \hat{\delta}}{\left(R + \hat{\delta} - \frac{\lambda}{\phi} \right)} \end{aligned} \tag{35}$$

By using (33) in (28):

$$\phi \left(1 - t_c\right) p F_K = \beta \left(R + \delta - \frac{\lambda}{\phi} \right) \tag{36}$$

Using (26) in (36) :

$$\phi \left(1 - t_c\right) p F_K = q \left(\phi - \gamma\right) \left(R + \delta - \frac{\lambda}{\phi} \right),$$

and by furthermore inserting (35) we obtain:

$$\phi \left(1 - t_c\right) p F_K = q \left(\phi - \frac{\phi t_c \hat{\delta}}{\left(R + \hat{\delta} - \frac{\lambda}{\phi} \right)} \right) \left(R + \delta - \frac{\lambda}{\phi} \right). \tag{37}$$

Finally use (33) in (30) and rearrange :

$$\begin{aligned} -\phi \left(1 - t_c\right) i - \lambda + \eta &= \alpha R \\ &\Updownarrow \text{ (ved (27))} \\ \lambda - \eta &= \phi \left(R - \left(1 - t_c\right) i \right) \end{aligned} \tag{38}$$

Now, note that (31) and (32) cannot be binding simultaneously, i.e. λ and η cannot both be greater than zero. Hence, we are left with three cases. $(\lambda > 0, \eta = 0)$, $(\lambda = 0, \eta > 0)$, $(\lambda = 0, \eta = 0)$. In the last case, neither Kuhn-Tucker conditions are binding.

Case I: $(\lambda > 0, \eta = 0)$: **Pure Debt finance** From (38) we have that

$$\lambda = \phi \left(R - \left(1 - t_c\right) i \right).$$

Substituting for λ in (37) we immediately obtain

$$\begin{aligned} \phi \left(1 - t_c\right) p F_K &= q \left(\phi - \frac{\phi t_c \hat{\delta}}{\left(R + \hat{\delta} - R + \left(1 - t_c\right) i \right)} \right) \left(R + \delta - \left(R - \left(1 - t_c\right) i \right) \right) \\ &\Updownarrow \\ \left(1 - t_c\right) p F_K &= q \left(1 - \frac{t_c \hat{\delta}}{\left(1 - t_c\right) i + \hat{\delta}} \right) \left(\delta + \left(1 - t_c\right) i \right). \end{aligned} \tag{39}$$

Case II: $\lambda = 0, \eta > 0$ **: Retained earnings** Using $\lambda = 0$ in (37) :

$$(1-t_c)\, pF_K = q\left(1-\frac{t_c\hat{\delta}}{\left(R+\hat{\delta}\right)}\right)(R+\delta)$$

$$\Updownarrow$$

$$(1-t_c)\, pF_K = q\left(1-\frac{t_c\hat{\delta}}{\left(\frac{i(1-t_r)}{(1-t_g)}+\hat{\delta}\right)}\right)\left(\frac{i(1-t_r)}{(1-t_g)}+\delta\right). \tag{40}$$

Case III: An intermediate case: In this situation (39) and (40) are *equivalent*, since $\lambda = 0, \eta = 0$ (by (38)) imply that

$$\frac{i(1-t_r)}{(1-t_g)} = (1-t_c)\, i.$$

C Stationary state restrictions

If we assume stationary state, what would the general restriction derived in appendix A look like? To answer this question we use the shadow prices evaluated in the stationary state. These are:

$$\gamma = \frac{\phi t_c\hat{\delta}}{\left(R+\hat{\delta}-\frac{\lambda}{\phi}\right)}$$

$$\beta = q(\phi-\gamma) = \phi q\left(1-\frac{t_c\hat{\delta}}{\left(R+\hat{\delta}-\frac{\lambda}{\phi}\right)}\right).$$

In the case of debt finance $\lambda > 0$ and (as we saw above) equal to $R-(1-t_c)i$. Hence,

$$\beta = \phi q\left(1-\frac{t_c\hat{\delta}}{i(1-t_c)+\hat{\delta}}\right)$$

$$\gamma = \phi\frac{t_c\hat{\delta}}{\left(i(1-t_c)+\hat{\delta}\right)}.$$

Inserted into the general restriction

$$K\frac{\beta}{\phi}+\frac{\gamma}{\phi}A-B \geq 0$$

we have the "stationary state" condition:

$$Kq\left(1-\frac{t_c\hat{\delta}}{i(1-t_c)+\hat{\delta}}\right)+\frac{t_c\hat{\delta}}{\left(i(1-t_c)+\hat{\delta}\right)}A-B \geq 0 \tag{41}$$

If $z \equiv \frac{t_c \hat{\delta}}{i(1-t_c)+\hat{\delta}}$ we have that

$$(1-z)\,Kq + zA - B \geq 0.$$

This is the restriction used in Schultz-Møller (1998). Finally one could be interested in the proper restriction on debt, in the stationary state, when $\hat{\delta} = \delta$. Since stationarity in particular implies that $\dot{K} = \dot{A} = 0$, it follows that $q\delta K = \delta A$ in this case. Using this in (41) yields

$$Kq - B \geq 0.$$

In this case the restriction on debt to use is (not too surprising) the value of the capital stock.